

NEW GENERAL INTEGRAL INEQUALITIES FOR (α, m)-GA-CONVEX FUNCTIONS VIA HADAMARD FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, the authors gives a new identity for Hadamard fractional integrals. By using of this identity, the authors obtains new estimates on generalization of Hadamard, Ostrowski and Simpson type inequalities for (α, m)-GA-convex function via Hadamard fractional integral.

1. INTRODUCTION

Let a real function f be defined on some nonempty interval I of real line \mathbb{R} . The function f is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Following inequalities are well known in the literature as Hermite-Hadamard inequality, Ostrowski inequality and Simpson inequality respectively:

Theorem 1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Theorem 2. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a mapping differentiable in I° , the interior of I , and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$, $x \in [a, b]$, then we the following inequality holds:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.

Theorem 3. Let $f : [a, b] \rightarrow \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^4.$$

The following definitions are well known in the literature.

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Definition 1. [10, 11]. A function $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2. [8]. Let $f : (0, b] \rightarrow \mathbb{R}, b > 0$, and $(\alpha, m) \in (0, 1]^2$. If

$$f(x^t y^{m(1-t)}) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all $x, y \in (0, b]$ and $t \in [0, 1]$, then f is said to be a (α, m) -GA-convex function.

Note that $(\alpha, m) \in \{(1, m), (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions: m -GA-convex, GA-convex, α -GA-convex (or GA- s -convex in the first sense, if we take s instead of α (see [19])).

We will now give definitions of the right-sided and left-sided Hadamard fractional integrals which are used throughout this paper.

Definition 3. [4]. Let $f \in L[a, b]$. The right-sided and left-sided Hadamard fractional integrals $J_{a+}^\theta f$ and $J_{b-}^\theta f$ of order $\theta > 0$ with $b > a \geq 0$ are defined by

$$J_{a+}^\theta f(x) = \frac{1}{\Gamma(\theta)} \int_a^x \left(\ln \frac{x}{t}\right)^{\theta-1} f(t) \frac{dt}{t}, \quad a < x < b$$

and

$$J_{b-}^\theta f(x) = \frac{1}{\Gamma(\theta)} \int_x^b \left(\ln \frac{t}{x}\right)^{\theta-1} f(t) \frac{dt}{t}, \quad a < x < b$$

respectively, where $\Gamma(\theta)$ is the Gamma function defined by $\Gamma(\theta) = \int_0^\infty e^{-t} t^{\theta-1} dt$.

In [20], İşcan represented Hermite-Hadamard's inequalities for GA-convex functions in fractional integral forms as follows:

Theorem 4. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If f is a GA-convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$f(\sqrt{ab}) \leq \frac{\Gamma(\theta+1)}{2(\ln \frac{b}{a})^\theta} \{J_{a+}^\theta f(b) + J_{b-}^\theta f(a)\} \leq \frac{f(a) + f(b)}{2}$$

with $\alpha > 0$.

In [20], İşcan gave the following identity for differentiable functions..

Lemma 1. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. Then for all $x \in [a, b]$, $\lambda \in [0, 1]$ and $\alpha > 0$

we have:

$$\begin{aligned}
I_f(x, \lambda, \theta, a, b) &= (1 - \lambda) \left[\ln^\theta \frac{x}{a} + \ln^\theta \frac{b}{x} \right] f(x) + \lambda \left[f(a) \ln^\theta \frac{x}{a} + f(b) \ln^\theta \frac{b}{x} \right] \\
&\quad - \Gamma(\theta + 1) [J_{x-}^\theta f(a) + J_{x+}^\theta f(b)] \\
&= a \left(\ln \frac{x}{a} \right)^{\theta+1} \int_0^1 (t^\theta - \lambda) \left(\frac{x}{a} \right)^t f'(x^t a^{1-t}) dt \\
&\quad - b \left(\ln \frac{b}{x} \right)^{\theta+1} \int_0^1 (t^\theta - \lambda) \left(\frac{x}{b} \right)^t f'(x^t b^{1-t}) dt.
\end{aligned}$$

In recent years, many authors have studied errors estimations for Hermite-Hadamard, Ostrowski and Simpson inequalities; for refinements, counterparts, generalization see [1, 2, 3, 5, 6, 7, 12, 13, 15, 16, 17, 18].

In this paper, new identity for fractional integrals have been defined. By using of this identity, we obtained a generalization of Hadamard, Ostrowski and Simpson type inequalities for (α, m) -GA-convex functions via Hadamard fractional integrals.

2. MAIN RESULTS

Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , the interior of I , throughout this section we will take

$$\begin{aligned}
K_f(\lambda, \theta, x^m, a^m, b^m) &= (1 - \lambda) m^\theta \left[\ln^\theta \frac{x}{a} + \ln^\theta \frac{b}{x} \right] f(x^m) \\
&\quad + \lambda m^\theta \left[f(a^m) \ln^\theta \frac{x}{a} + f(b^m) \ln^\theta \frac{b}{x} \right] \\
&\quad - \Gamma(\theta + 1) [J_{x^m-}^\theta f(a^m) + J_{x^m+}^\theta f(b^m)]
\end{aligned}$$

where $a, b \in I$ with $a < b$, $x \in [a, b]$, $\lambda \in [0, 1]$, $\theta > 0$ and Γ is Euler Gamma function.

Similarly to Lemma 1, we can prove the following lemma.

Lemma 2. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a^m, b^m]$, where $a^m, b \in I$ with $a < b$ and $m \in (0, 1]$. Then for all $x \in [a, b]$, $\lambda \in [0, 1]$ and $\theta > 0$ we have:

$$\begin{aligned}
K_f(\lambda, \theta, x^m, a^m, b^m) &= m^{\theta+1} a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \int_0^1 (t^\theta - \lambda) \left(\frac{x}{a} \right)^{mt} f'(x^{mt} a^{m(1-t)}) dt \\
&\quad - m^{\theta+1} b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \int_0^1 (t^\theta - \lambda) \left(\frac{x}{b} \right)^{mt} f'(x^{mt} b^{m(1-t)}) dt.
\end{aligned}$$

Theorem 5. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a^m, b^m]$, where $a^m, b \in I^\circ$ with $a < b$ and $m \in (0, 1]$. If $|f'|^q$ is (α, m) -GA-convex on $[a^m, b]$ for some fixed $q \geq 1$, $x \in [a, b]$, $\lambda \in [0, 1]$ and $\theta > 0$ then the

following inequality for fractional integrals holds

$$\begin{aligned}
|K_f(\lambda, \theta, x^m, a^m, b^m)| &\leq m^{\theta+1} C_o(\theta, \lambda)^{1-\frac{1}{q}} \\
&\times \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \left(\frac{|f'(x^m)|^q C_1(x, \theta, \lambda, q, m, \alpha)}{+m |f'(a)|^q C_2(x, \theta, \lambda, q, m, \alpha)} \right)^{\frac{1}{q}} \right. \\
&\left. + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \left(\frac{|f'(x^m)|^q C_3(x, \theta, \lambda, q, m, \alpha)}{+m |f'(b)|^q C_4(x, \theta, \lambda, q, m, \alpha)} \right)^{\frac{1}{q}} \right\} \quad (2.1)
\end{aligned}$$

where

$$\begin{aligned}
C_o(\theta, \lambda) &= \frac{2\theta\lambda^{1+\frac{1}{\theta}} + 1}{\theta + 1} - \lambda, \\
C_1(x, \theta, \lambda, q, m, \alpha) &= \int_0^1 |t^\theta - \lambda| \left(\frac{x}{a} \right)^{qmt} t^\alpha dt, \\
C_2(x, \theta, \lambda, q, m, \alpha) &= \int_0^1 |t^\theta - \lambda| \left(\frac{x}{a} \right)^{qmt} (1 - t^\alpha) dt, \\
C_3(x, \theta, \lambda, q, m, \alpha) &= \int_0^1 |t^\theta - \lambda| \left(\frac{x}{b} \right)^{qmt} t^\alpha dt, \\
C_4(x, \theta, \lambda, q, m, \alpha) &= \int_0^1 |t^\theta - \lambda| \left(\frac{x}{b} \right)^{qmt} (1 - t^\alpha) dt.
\end{aligned}$$

Proof. From Lemma 2, property of the modulus and using the power-mean inequality we have

$$\begin{aligned}
|K_f(\lambda, \theta, x^m, a^m, b^m)| &\leq m^{\theta+1} \left(\int_0^1 |t^\theta - \lambda| dt \right)^{1-\frac{1}{q}} \\
&\times \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \left(\int_0^1 |t^\theta - \lambda| \left(\frac{x}{a} \right)^{qmt} |f'(x^{mt} a^{m(1-t)})|^q dt \right)^{\frac{1}{q}} \right. \\
&\left. + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \left(\int_0^1 |t^\theta - \lambda| \left(\frac{x}{b} \right)^{qmt} |f'(x^{mt} b^{m(1-t)})|^q dt \right)^{\frac{1}{q}} \right\}. \quad (2.2)
\end{aligned}$$

Since $|f'|^q$ is (α, m) -GA-convex on $[a^m, b]$, for all $t \in [0, 1]$

$$\left| f'(x^{mt} a^{m(1-t)}) \right|^q \leq t^\alpha |f'(x^m)|^q + m(1-t^\alpha) |f'(a)|^q, \quad (2.3)$$

$$\left| f'(x^{mt} b^{m(1-t)}) \right|^q \leq t^\alpha |f'(x^m)|^q + m(1-t^\alpha) |f'(b)|^q. \quad (2.4)$$

By a simple computation

$$\begin{aligned}
\int_0^1 |t^\theta - \lambda| dt &= \int_0^{\lambda^{1/\theta}} (\lambda - t^\theta) dt + \int_{\lambda^{1/\theta}}^1 (t^\theta - \lambda) dt \\
&= \frac{2\theta\lambda^{1+\frac{1}{\theta}} + 1}{\theta + 1} - \lambda. \quad (2.5)
\end{aligned}$$

If we use (2.3), (2.4) and (2.5) in (2.2), we obtain (2.1). This completes the proof. \square

Corollary 1. *Under the assumptions of Theorem 5 with $q = 1$, the inequality (2.1) reduced to the following inequality*

$$\begin{aligned} K_f(\lambda, \theta, x^m, a^m, b^m) &\leq m^{\theta+1} \\ &\times \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \begin{pmatrix} |f'(x^m)| C_1(x, \theta, \lambda, 1, m, \alpha) \\ + m |f'(a)| C_2(x, \theta, \lambda, 1, m, \alpha) \end{pmatrix} \right. \\ &\left. + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \begin{pmatrix} |f'(x^m)| C_3(x, \theta, \lambda, 1, m, \alpha) \\ + m |f'(b)| C_4(x, \theta, \lambda, 1, m, \alpha) \end{pmatrix} \right\} \end{aligned}$$

Corollary 2. *Under the assumptions of Theorem 5 with $x = \sqrt{ab}$, $\lambda = \frac{1}{3}$ from the inequality (2.1) we get the following Simpson type inequality for fractional integrals*

$$\begin{aligned} &\frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left(\frac{1}{3}, \theta, (\sqrt{ab})^m, a^m, b^m \right) \right| = \left| \frac{1}{6} [f(a^m) + 4f((\sqrt{ab})^m) + f(b^m)] \right. \\ &- \frac{2^{\theta-1} \Gamma(\theta+1)}{(m \ln \frac{b}{a})^\theta} [J_{(\sqrt{ab})^m-}^\theta f(a^m) + J_{(\sqrt{ab})^m+}^\theta f(b^m)] \left. \right| \leq \frac{m \ln \frac{b}{a}}{4} C_0^{1-\frac{1}{q}} \left(\theta, \frac{1}{3} \right) \\ &\times \left\{ a^m \left[\begin{array}{l} |f'((\sqrt{ab})^m)|^q C_1(\sqrt{ab}, \theta, \frac{1}{3}, q, m, \alpha) \\ + m |f'(a)|^q C_2(\sqrt{ab}, \theta, \frac{1}{3}, q, m, \alpha) \end{array} \right]^{\frac{1}{q}} \right. \\ &\left. + b^m \left[\begin{array}{l} |f'((\sqrt{ab})^m)|^q C_3(\sqrt{ab}, \theta, \frac{1}{3}, q, m, \alpha) \\ + m |f'(b)|^q C_4(\sqrt{ab}, \theta, \frac{1}{3}, q, m, \alpha) \end{array} \right]^{\frac{1}{q}} \right\} \end{aligned}$$

Corollary 3. *Under the assumptions of Theorem 5 with $x = \sqrt{ab}$, $\lambda = 0$ from the inequality (2.1) we get the following midpoint-type inequality for fractional integrals*

$$\begin{aligned} &\frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f(0, \theta, (\sqrt{ab})^m, a^m, b^m) \right| \\ &= \left| f((\sqrt{ab})^m) - \frac{2^{\theta-1} \Gamma(\theta+1)}{(m \ln \frac{b}{a})^\theta} [J_{(\sqrt{ab})^m-}^\theta f(a^m) + J_{(\sqrt{ab})^m+}^\theta f(b^m)] \right| \\ &\leq \frac{m \ln \frac{b}{a}}{4} \left(\frac{1}{\theta+1} \right)^{1-\frac{1}{q}} \left\{ a^m \left[\begin{array}{l} |f'((\sqrt{ab})^m)|^q C_1(\sqrt{ab}, \theta, 0, q, m, \alpha) \\ + m |f'(a)|^q C_2(\sqrt{ab}, \theta, 0, q, m, \alpha) \end{array} \right]^{\frac{1}{q}} \right. \\ &\left. + b^m \left[\begin{array}{l} |f'((\sqrt{ab})^m)|^q C_3(\sqrt{ab}, \theta, 0, q, m, \alpha) \\ + m |f'(b)|^q C_4(\sqrt{ab}, \theta, 0, q, m, \alpha) \end{array} \right]^{\frac{1}{q}} \right\} \end{aligned}$$

Remark 1. *If we take $\theta = 1$, $m = 1$ in Corollary 3 we have the following midpoint-type inequality for α -GA-convex function (or GA-s-convex function in the first*

sense), which is the same with the inequality (9) of Theorem 3.4.b. in [19],

$$\begin{aligned} & \left| f(\sqrt{ab}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \\ & \leq \ln \frac{b}{a} \left(\frac{1}{2} \right)^{3-\frac{1}{q}} \left\{ a \left[\left| f'(\sqrt{ab}) \right|^q C_1(\sqrt{ab}, 1, 0, q, 1, \alpha) \right. \right. \\ & \quad \left. \left. + |f'(a)|^q C_2(\sqrt{ab}, 1, 0, q, 1, \alpha) \right] \right]^{\frac{1}{q}} \\ & \quad + b \left[\left| f'(\sqrt{ab}) \right|^q C_3(\sqrt{ab}, 1, 0, q, 1, \alpha) \right. \\ & \quad \left. + |f'(b)|^q C_4(\sqrt{ab}, 1, 0, q, 1, \alpha) \right] \right]^{\frac{1}{q}} \Bigg\}. \end{aligned}$$

Remark 2. If we take $\theta = 1$, $m = 1$, $\alpha = 1$ in Corollary 3 we have the following midpoint-type inequality for GA-convex function, which is the same with the inequality (13) of Corollary 3.5 in [19],

$$\begin{aligned} & \left| f(\sqrt{ab}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \\ & \leq \ln \frac{b}{a} \left(\frac{1}{2} \right)^{3-\frac{1}{q}} \left\{ a \left[\left| f'(\sqrt{ab}) \right|^q C_1(\sqrt{ab}, 1, 0, q, 1, 1) \right. \right. \\ & \quad \left. \left. + |f'(a)|^q C_2(\sqrt{ab}, 1, 0, q, 1, 1) \right] \right]^{\frac{1}{q}} \\ & \quad + b \left[\left| f'(\sqrt{ab}) \right|^q C_3(\sqrt{ab}, 1, 0, q, 1, 1) \right. \\ & \quad \left. + |f'(b)|^q C_4(\sqrt{ab}, 1, 0, q, 1, 1) \right] \right]^{\frac{1}{q}} \Bigg\}. \end{aligned}$$

Corollary 4. Under the assumptions of Theorem 5 with $x = \sqrt{ab}$, $\lambda = 1$ from the inequality (2.1) we get the following trapezoid-type inequality for fractional integrals

$$\begin{aligned} & \frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left(1, \theta, (\sqrt{ab})^m, a^m, b^m \right) \right| \\ & = \left| \frac{f(a^m) + f(b^m)}{2} - \frac{2^{\theta-1} \Gamma(\theta+1)}{(m \ln \frac{b}{a})^\theta} \left[J_{(\sqrt{ab})^m-}^\theta f(a^m) + J_{(\sqrt{ab})^m+}^\theta f(b^m) \right] \right| \\ & \leq \frac{m \ln \frac{b}{a}}{4} \left(\frac{\theta}{\theta+1} \right)^{1-\frac{1}{q}} \left\{ a^m \left[\left| f'((\sqrt{ab})^m) \right|^q C_1(\sqrt{ab}, \theta, 1, q, m, \alpha) \right. \right. \\ & \quad \left. \left. + m |f'(a)|^q C_2(\sqrt{ab}, \theta, 1, q, m, \alpha) \right] \right]^{\frac{1}{q}} \\ & \quad + b^m \left[\left| f'((\sqrt{ab})^m) \right|^q C_3(\sqrt{ab}, \theta, 1, q, m, \alpha) \right. \\ & \quad \left. + m |f'(b)|^q C_4(\sqrt{ab}, \theta, 1, q, m, \alpha) \right] \right]^{\frac{1}{q}} \Bigg\}. \end{aligned}$$

Corollary 5. Let the assumptions of Theorem 5 hold. If $|f'(u)| \leq M$ for all $u \in [a^m, b]$ and $\lambda = 0$, then from the inequality (2.1) we get the following Ostrowski

type inequality for fractional integrals

$$\begin{aligned} & \left| \left[\left(\ln \frac{x}{a} \right)^\theta + \left(\ln \frac{b}{x} \right)^\theta \right] f(x^m) - \frac{\Gamma(\theta+1)}{m^\theta} [J_{x^m-}^\theta f(a^m) + J_{x^m+}^\theta f(b^m)] \right| \\ & \leq \frac{mM}{(\theta+1)^{1-\frac{1}{q}}} \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \begin{pmatrix} C_1(x, \theta, 0, q, m, \alpha) \\ +mC_2(x, \theta, 0, q, m, \alpha) \end{pmatrix}^{\frac{1}{q}} \right. \\ & \quad \left. + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \begin{pmatrix} C_3(x, \theta, \lambda, q, m, \alpha) \\ +mC_4(x, \theta, \lambda, q, m, \alpha) \end{pmatrix}^{\frac{1}{q}} \right\} \end{aligned}$$

for each $x \in [a, b]$.

Theorem 6. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a^m, b^m]$, where $a^m, b \in I^\circ$ with $a < b$ and $m \in (0, 1]$. If $|f'|^q$ is (α, m) -GA-convex on $[a^m, b]$ for some fixed $q > 1$, $x \in [a, b]$, $\lambda \in [0, 1]$ and $\theta > 0$ then the following inequality for fractional integrals holds

$$\begin{aligned} & |K_f(\lambda, \theta, x^m, a^m, b^m)| \leq m^{\theta+1} R_0^{\frac{1}{p}}(\theta, \lambda, p) \\ & \times \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \begin{pmatrix} |f'(x^m)|^q R_1(x, q, m, \alpha) \\ +m|f'(a)|^q R_2(x, q, m, \alpha) \end{pmatrix}^{\frac{1}{q}} \right. \\ & \quad \left. + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \begin{pmatrix} |f'(x^m)|^q R_3(x, q, m, \alpha) \\ +m|f'(b)|^q R_4(x, q, m, \alpha) \end{pmatrix}^{\frac{1}{q}} \right\} \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} R_0(\theta, \lambda, p) &= \int_0^1 |t^\theta - \lambda|^p dt \\ &= \begin{cases} \frac{1}{\theta p + 1}, & \lambda = 0 \\ \left\{ \frac{\lambda^{(\theta p + 1)/\theta}}{\theta} \beta\left(\frac{1}{\theta}, p + 1\right) + \frac{(1-\lambda)^{p+1}}{\theta(p+1)} \right. \\ \quad \left. \times {}_2F_1\left(1 - \frac{1}{\theta}, 1; p + 2; 1 - \lambda\right) \right\} \\ \frac{1}{\theta} \beta\left(\frac{1}{\theta}, p + 1\right), & \lambda = 1 \end{cases}, \\ R_1(x, q, m, \alpha) &= \int_0^1 \left(\frac{x}{a}\right)^{mqt} t^\alpha dt, \\ R_2(x, q, m, \alpha) &= \int_0^1 \left(\frac{x}{a}\right)^{mqt} (1 - t^\alpha) dt, \\ R_3(x, q, m, \alpha) &= \int_0^1 \left(\frac{x}{b}\right)^{mqt} t^\alpha dt, \\ R_4(x, q, m, \alpha) &= \int_0^1 \left(\frac{x}{b}\right)^{mqt} (1 - t^\alpha) dt, \end{aligned}$$

${}_2F_1$ is hypergeometrical function defined by

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \\ c &> b > 0, |z| < 1 \text{ (see [4])}, \end{aligned}$$

β is beta function defined by

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0,$$

$$\text{and } \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. From Lemma 2, property of the modulus and using the Hölder inequality we have

$$\begin{aligned} |K_f(\lambda, \theta, x^m, a^m, b^m)| &\leq m^{\theta+1} \left(\int_0^1 |t^\theta - \lambda|^p dt \right)^{\frac{1}{p}} \\ &\times \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \left(\int_0^1 \left(\frac{x}{a} \right)^{qmt} |f'(x^{mt} a^{m(1-t)})|^q dt \right)^{\frac{1}{q}} \right. \\ &\quad \left. + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \left(\int_0^1 \left(\frac{x}{b} \right)^{qmt} |f'(x^{mt} b^{m(1-t)})|^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned} \quad (2.7)$$

By a simple computation

$$\begin{aligned} R_0(\theta, \lambda, p) &= \int_0^1 |t^\theta - \lambda|^p dt \\ &= \begin{cases} \frac{1}{\theta p + 1}, & \lambda = 0 \\ \left\{ \frac{\lambda^{(\theta p + 1)/\theta}}{\theta} \beta\left(\frac{1}{\theta}, p+1\right) + \frac{(1-\lambda)^{p+1}}{\theta(p+1)} \right. \\ \quad \left. \times {}_2F_1\left(1 - \frac{1}{\theta}, 1; p+2; 1-\lambda\right) \right\}, & 0 < \lambda < 1 \\ \frac{1}{\theta} \beta\left(\frac{1}{\theta}, p+1\right), & \lambda = 1 \end{cases}, \end{aligned} \quad (2.8)$$

Since $|f'|^q$ is (α, m) -GA-convex on $[a^m, b]$, for all $t \in [0, 1]$, if we use (2.3), (2.4) and (2.8) in (2.7), we obtain (2.6). This completes the proof. \square

Corollary 6. *Under the assumptions of Theorem 6 with $x = \sqrt{ab}$, $\lambda = \frac{1}{3}$ from the inequality (2.6) we get the following Simpson type inequality for fractional integrals*

$$\begin{aligned} &\frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f\left(\frac{1}{3}, \theta, (\sqrt{ab})^m, a^m, b^m\right) \right| = \left| \frac{1}{6} [f(a^m) + 4f((\sqrt{ab})^m) + f(b^m)] \right. \\ &\quad \left. - \frac{2^{\theta-1}\Gamma(\theta+1)}{(m \ln \frac{b}{a})^\theta} [J_{(\sqrt{ab})^m}^\theta f(a^m) + J_{(\sqrt{ab})^m}^\theta f(b^m)] \right| \leq \frac{m \ln \frac{b}{a}}{4} R_0^{\frac{1}{p}}\left(\theta, \frac{1}{3}, p\right) \\ &\quad \times \left\{ a^m \left[\begin{array}{l} |f'((\sqrt{ab})^m)|^q R_1(\sqrt{ab}, q, m, \alpha) \\ + m |f'(a)|^q R_2(\sqrt{ab}, q, m, \alpha) \end{array} \right]^{\frac{1}{q}} \right. \\ &\quad \left. + b^m \left[\begin{array}{l} |f'((\sqrt{ab})^m)|^q R_3(\sqrt{ab}, q, m, \alpha) \\ + m |f'(b)|^q R_4(\sqrt{ab}, q, m, \alpha) \end{array} \right]^{\frac{1}{q}} \right\} \end{aligned}$$

Corollary 7. *Under the assumptions of Theorem 6 with $x = \sqrt{ab}$, $\lambda = 0$ from the inequality (2.6) we get the following midpoint-type inequality for fractional integrals*

$$\begin{aligned}
& \frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left(0, \theta, (\sqrt{ab})^m, a^m, b^m \right) \right| \\
&= \left| f \left((\sqrt{ab})^m \right) - \frac{2^{\theta-1} \Gamma(\theta+1)}{(m \ln \frac{b}{a})^\theta} \left[J_{(\sqrt{ab})^m-}^\theta f(a^m) + J_{(\sqrt{ab})^m+}^\theta f(b^m) \right] \right| \\
&\leq \frac{m \ln \frac{b}{a}}{4} \left(\frac{1}{\theta p + 1} \right)^{\frac{1}{p}} \left\{ a^m \left[\begin{aligned} & \left| f' \left((\sqrt{ab})^m \right) \right|^q R_1 \left(\sqrt{ab}, q, m, \alpha \right) \\ & + m |f'(a)|^q R_2 \left(\sqrt{ab}, q, m, \alpha \right) \end{aligned} \right]^{\frac{1}{q}} \right. \\
&\quad \left. + b^m \left[\begin{aligned} & \left| f' \left((\sqrt{ab})^m \right) \right|^q R_3 \left(\sqrt{ab}, q, m, \alpha \right) \\ & + m |f'(b)|^q R_4 \left(\sqrt{ab}, q, m, \alpha \right) \end{aligned} \right]^{\frac{1}{q}} \right\}
\end{aligned}$$

Remark 3. If we take $\theta = 1$, $m = 1$, $p = \frac{q}{q-1}$ in Corollary 7 we have the following midpoint-type inequality for α -GA-convex function (or GA-s-convex function in the first sense), which is the same with the inequality (17) of Theorem 3.7.b. in [19],

$$\begin{aligned}
& \left| f(\sqrt{ab}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \\
&\leq \frac{\ln \frac{b}{a}}{4} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \left\{ a \left[\begin{aligned} & \left| f'(\sqrt{ab}) \right|^q R_1(\sqrt{ab}, q, 1, \alpha) \\ & + |f'(a)|^q R_2(\sqrt{ab}, q, 1, \alpha) \end{aligned} \right]^{\frac{1}{q}} \right. \\
&\quad \left. + b \left[\begin{aligned} & \left| f'(\sqrt{ab}) \right|^q R_3(\sqrt{ab}, q, 1, \alpha) \\ & + |f'(b)|^q R_4(\sqrt{ab}, q, 1, \alpha) \end{aligned} \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

Remark 4. If we take $\theta = 1$, $m = 1$, $\alpha = 1$, $p = \frac{q}{q-1}$ in Corollary 7 we have the following midpoint-type inequality for GA-convex function, which is the same with the inequality (21) of Corollary 3.8 in [19],

$$\begin{aligned}
& \left| f(\sqrt{ab}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right| \\
&\leq \frac{\ln \frac{b}{a}}{4} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \left\{ a \left[\begin{aligned} & \left| f'(\sqrt{ab}) \right|^q R_1(\sqrt{ab}, q, 1, 1) \\ & + |f'(a)|^q R_2(\sqrt{ab}, q, 1, 1) \end{aligned} \right]^{\frac{1}{q}} \right. \\
&\quad \left. + b \left[\begin{aligned} & \left| f'(\sqrt{ab}) \right|^q R_3(\sqrt{ab}, q, 1, 1) \\ & + |f'(b)|^q R_4(\sqrt{ab}, q, 1, 1) \end{aligned} \right]^{\frac{1}{q}} \right\} ..
\end{aligned}$$

Corollary 8. *Under the assumptions of Theorem 6 with $x = \sqrt{ab}$, $\lambda = 1$ from the inequality (2.6) we get the following trepezoid-type inequality for fractional integrals*

$$\begin{aligned}
& \frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left(1, \theta, \left(\sqrt{ab} \right)^m, a^m, b^m \right) \right| \\
&= \left| \frac{f(a^m) + f(b^m)}{2} - \frac{2^{\theta-1} \Gamma(\theta+1)}{(m \ln \frac{b}{a})^\theta} \left[J_{(\sqrt{ab})^m-}^\theta f(a^m) + J_{(\sqrt{ab})^m+}^\theta f(b^m) \right] \right| \\
&\leq \frac{m \ln \frac{b}{a}}{4} \left(\frac{1}{\theta} \beta \left(\frac{1}{\theta}, p+1 \right) \right)^{\frac{1}{p}} \left\{ a^m \left[\begin{array}{c} |f'((\sqrt{ab})^m)|^q R_1(\sqrt{ab}, q, m, \alpha) \\ + m |f'(a)|^q R_2(\sqrt{ab}, q, m, \alpha) \end{array} \right]^{\frac{1}{q}} \right. \\
&\quad \left. + b^m \left[\begin{array}{c} |f'(x^m)|^q R_3(\sqrt{ab}, q, m, \alpha) \\ + m |f'(b)|^q R_4(\sqrt{ab}, q, m, \alpha) \end{array} \right]^{\frac{1}{q}} \right\}
\end{aligned}$$

Corollary 9. *Let the assumptions of Theorem 6 hold. If $|f'(u)| \leq M$ for all $u \in [a^m, b]$ and $\lambda = 0$, then from the inequality (2.6) we get the following Ostrowski type inequality for fractional integrals*

$$\begin{aligned}
& \left| \left[\left(\ln \frac{x}{a} \right)^\theta + \left(\ln \frac{b}{x} \right)^\theta \right] f(x^m) - \frac{\Gamma(\theta+1)}{m^\theta} \left[J_{x^m-}^\theta f(a^m) + J_{x^m+}^\theta f(b^m) \right] \right| \\
&\leq \frac{mM}{(\theta p + 1)^{\frac{1}{p}}} \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \left(\begin{array}{c} R_1(x, q, m, \alpha) \\ + m R_2(x, q, m, \alpha) \end{array} \right)^{\frac{1}{q}} \right. \\
&\quad \left. + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \left(\begin{array}{c} R_3(x, q, m, \alpha) \\ + R_4(x, q, m, \alpha) \end{array} \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

for each $x \in [a, b]$

Theorem 7. *Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a^m, b^m]$, where $a^m, b \in I^\circ$ with $a < b$ and $m \in (0, 1]$. If $|f'|^q$ is (α, m) -GA-convex on $[a^m, b]$ for some fixed $q > 1$, $x \in [a, b]$, $\lambda \in [0, 1]$ and $\theta > 0$ then the following inequality for fractional integrals holds*

$$\begin{aligned}
& |K_f(\lambda, \theta, x^m, a^m, b^m)| \leq m^{\theta+1} \\
& \times \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} T_1^{\frac{1}{p}}(x, \theta, \lambda, p, m) \left(\frac{|f'(x^m)|^q + m\alpha |f'(a)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right. \\
& \left. + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} T_2^{\frac{1}{p}}(x, \theta, \lambda, p, m) \left(\frac{|f'(x^m)|^q + m\alpha |f'(b)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right\} \quad (2.9)
\end{aligned}$$

where

$$\begin{aligned}
T_1(x, \theta, \lambda, p, m) &= \int_0^1 |t^\theta - \lambda|^p \left(\frac{x}{a} \right)^{mpt} dt, \\
T_2(x, \theta, \lambda, p, m) &= \int_0^1 |t^\theta - \lambda|^p \left(\frac{x}{b} \right)^{mpt} dt,
\end{aligned}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $|f'|^q$ is (α, m) -GA-convex on $[a^m, b]$, for all $t \in [0, 1]$, if we use (2.3), (2.4)

$$\begin{aligned} \int_0^1 \left| f' \left(x^{mt} a^{m(1-t)} \right) \right|^q dt &\leq \int_0^1 t^\alpha |f'(x^m)|^q + m(1-t^\alpha) |f'(a)|^q dt \\ &= \frac{|f'(x^m)|^q + m\alpha |f'(a)|^q}{\alpha + 1}, \end{aligned} \quad (2.10)$$

$$\begin{aligned} \int_0^1 \left| f' \left(x^{mt} b^{m(1-t)} \right) \right|^q dt &\leq \int_0^1 t^\alpha |f'(x^m)|^q + m(1-t^\alpha) |f'(b)|^q dt \\ &= \frac{|f'(x^m)|^q + m\alpha |f'(b)|^q}{\alpha + 1}. \end{aligned} \quad (2.11)$$

From Lemma 2, property of the modulus, (2.10), (2.11) and using the Hölder inequality, we have

$$\begin{aligned} &|K_f(\lambda, \theta, x^m, a^m, b^m)| \leq m^{\theta+1} \\ &\times \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \left(\int_0^1 |t^\theta - \lambda|^p \left(\frac{x}{a} \right)^{mpt} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f' \left(x^{mt} a^{m(1-t)} \right) \right|^q dt \right)^{\frac{1}{q}} \right. \\ &\quad \left. + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \left(\int_0^1 |t^\theta - \lambda|^p \left(\frac{x}{b} \right)^{mpt} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f' \left(x^{mt} b^{m(1-t)} \right) \right|^q dt \right)^{\frac{1}{q}} \right\} \\ &\leq m^{\theta+1} \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \left(\int_0^1 |t^\theta - \lambda|^p \left(\frac{x}{a} \right)^{mpt} dt \right)^{\frac{1}{p}} \left(\frac{|f'(x^m)|^q + m\alpha |f'(a)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right. \\ &\quad \left. + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \left(\int_0^1 |t^\theta - \lambda|^p \left(\frac{x}{b} \right)^{mpt} dt \right)^{\frac{1}{p}} \left(\frac{|f'(x^m)|^q + m\alpha |f'(b)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right\} \end{aligned}$$

This completes the proof. \square

Corollary 10. Under the assumptions of Theorem 7 with $x = \sqrt{ab}$, $\lambda = \frac{1}{3}$ from the inequality (2.9) we get the following Simpson type inequality for fractional integrals

$$\begin{aligned} &\frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left(\frac{1}{3}, \theta, (\sqrt{ab})^m, a^m, b^m \right) \right| = \left| \frac{1}{6} \left[f(a^m) + 4f((\sqrt{ab})^m) + f(b^m) \right] \right. \\ &\quad \left. - \frac{2^{\theta-1} \Gamma(\theta+1)}{(m \ln \frac{b}{a})^\theta} \left[J_{(\sqrt{ab})^m}^\theta f(a^m) + J_{(\sqrt{ab})^m}^\theta f(b^m) \right] \right| \leq \frac{m \ln \frac{b}{a}}{4} \\ &\times \left\{ a^m T_1^{\frac{1}{p}} \left(\sqrt{ab}, \theta, \frac{1}{3}, p, m \right) \left(\frac{|f'((\sqrt{ab})^m)|^q + m\alpha |f'(a)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right. \\ &\quad \left. + b^m T_2^{\frac{1}{p}} \left(\sqrt{ab}, \theta, \frac{1}{3}, p, m \right) \left(\frac{|f'((\sqrt{ab})^m)|^q + m\alpha |f'(b)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right\} \end{aligned}$$

Corollary 11. *Under the assumptions of Theorem 7 with $x = \sqrt{ab}$, $\lambda = 0$ from the inequality (2.9) we get the following midpoint-type inequality for fractional integrals*

$$\begin{aligned}
& \frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left(0, \theta, \left(\sqrt{ab} \right)^m, a^m, b^m \right) \right| \\
&= \left| f \left(\left(\sqrt{ab} \right)^m \right) - \frac{2^{\theta-1} \Gamma(\theta+1)}{(m \ln \frac{b}{a})^\theta} \left[J_{(\sqrt{ab})^m-}^\theta f(a^m) + J_{(\sqrt{ab})^m+}^\theta f(b^m) \right] \right| \\
&\leq \frac{m \ln \frac{b}{a}}{4} \left\{ a^m T_1^{\frac{1}{p}} \left(\sqrt{ab}, \theta, 0, p, m \right) \left(\frac{\left| f' \left(\left(\sqrt{ab} \right)^m \right) \right|^q + m\alpha |f'(a)|^q}{\alpha+1} \right)^{\frac{1}{q}} \right. \\
&\quad \left. + b^m T_2^{\frac{1}{p}} \left(\sqrt{ab}, \theta, 0, p, m \right) \left(\frac{\left| f' \left(\left(\sqrt{ab} \right)^m \right) \right|^q + m\alpha |f'(b)|^q}{\alpha+1} \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

Corollary 12. *Under the assumptions of Theorem 7 with $x = \sqrt{ab}$, $\lambda = 1$ from the inequality (2.9) we get the following trapezoid-type inequality for fractional integrals*

$$\begin{aligned}
& \frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left(1, \theta, \left(\sqrt{ab} \right)^m, a^m, b^m \right) \right| \\
&= \left| \frac{f(a^m) + f(b^m)}{2} - \frac{2^{\theta-1} \Gamma(\theta+1)}{(m \ln \frac{b}{a})^\theta} \left[J_{(\sqrt{ab})^m-}^\theta f(a^m) + J_{(\sqrt{ab})^m+}^\theta f(b^m) \right] \right| \\
&\leq \frac{m \ln \frac{b}{a}}{4} \left\{ a^m T_1^{\frac{1}{p}} \left(\sqrt{ab}, \theta, 1, p, m \right) \left(\frac{\left| f' \left(\left(\sqrt{ab} \right)^m \right) \right|^q + m\alpha |f'(a)|^q}{\alpha+1} \right)^{\frac{1}{q}} \right. \\
&\quad \left. + b^m T_2^{\frac{1}{p}} \left(\sqrt{ab}, \theta, 1, p, m \right) \left(\frac{\left| f' \left(\left(\sqrt{ab} \right)^m \right) \right|^q + m\alpha |f'(b)|^q}{\alpha+1} \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

Corollary 13. *Let the assumptions of Theorem 7 hold. If $|f'(u)| \leq M$ for all $u \in [a^m, b]$ and $\lambda = 0$, then from the inequality (2.9) we get the following Ostrowski type inequality for fractional integrals*

$$\begin{aligned}
& \left| \left[\left(\ln \frac{x}{a} \right)^\theta + \left(\ln \frac{b}{x} \right)^\theta \right] f(x^m) - \frac{\Gamma(\theta+1)}{m^\theta} \left[J_{x^m-}^\theta f(a^m) + J_{x^m+}^\theta f(b^m) \right] \right| \\
&\leq mM \left(\frac{1+m\alpha}{\alpha+1} \right)^{\frac{1}{q}} \\
&\quad \times \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} T_1^{\frac{1}{p}}(x, \theta, 0, p, m) + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} T_2^{\frac{1}{p}}(x, \theta, 0, p, m) \right\}
\end{aligned}$$

for each $x \in [a, b]$

Theorem 8. *Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a^m, b^m]$, where $a^m, b \in I^\circ$ with $a < b$ and $m \in (0, 1]$. If $|f'|^q$ is (α, m) -GA-convex on $[a^m, b]$ for some fixed $q > 1$, $x \in [a, b]$, $\lambda \in [0, 1]$ and $\theta > 0$ then the*

following inequality for fractional integrals holds

$$\begin{aligned}
& |K_f(\lambda, \theta, x^m, a^m, b^m)| \leq m^{\theta+1} \\
& \times \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} V_3^{\frac{1}{p}} \left[\begin{array}{c} V_1(\theta, \lambda, \alpha, q) |f'(x^m)|^q \\ + m V_2(\theta, \lambda, \alpha, q) |f'(a)|^q \end{array} \right]^{\frac{1}{q}} \right. \\
& \left. + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} V_4^{\frac{1}{p}} \left[\begin{array}{c} V_1(\theta, \lambda, \alpha, q) |f'(x^m)|^q \\ + m V_2(\theta, \lambda, \alpha, q) |f'(b)|^q \end{array} \right]^{\frac{1}{q}} \right\} \quad (2.12)
\end{aligned}$$

where

$$\begin{aligned}
V_1(\theta, \lambda, \alpha, q) &= \int_0^1 |t^\theta - \lambda|^q t^\alpha dt \\
&= \begin{cases} \frac{1}{\theta q + \alpha + 1} & , \quad \lambda = 0 \\ \left\{ \frac{\lambda^{(\theta q + \alpha + 1)/\theta}}{\theta} \beta\left(\frac{\alpha+1}{\theta}, q+1\right) + \frac{(1-\lambda)^{q+1}}{\theta(q+1)} \right. \\ \quad \times {}_2F_1\left(1 - \frac{\alpha+1}{\theta}, 1; q+2; 1-\lambda\right) \left. \right\} & , \quad 0 < \lambda < 1 \\ \frac{1}{\theta} \beta\left(\frac{\alpha+1}{\theta}, q+1\right) & , \quad \lambda = 1 \end{cases} \quad (2.13)
\end{aligned}$$

$$\begin{aligned}
V_2(\theta, \lambda, \alpha, q) &= \int_0^1 |t^\theta - \lambda|^q (1-t^\alpha) dt \\
&= \begin{cases} \frac{1}{\theta q + 1} - \frac{1}{\theta q + \alpha + 1} & , \quad \lambda = 0 \\ \left\{ \frac{\lambda^{(\theta q + 1)/\theta}}{\theta} \beta\left(\frac{1}{\theta}, q+1\right) - \frac{\lambda^{(\theta q + \alpha + 1)/\theta}}{\theta} \beta\left(\frac{\alpha+1}{\theta}, q+1\right) \right. \\ \quad + \frac{(1-\lambda)^{q+1}}{\theta(q+1)} \left(\begin{array}{c} {}_2F_1\left(1 - \frac{1}{\theta}, 1; q+2; 1-\lambda\right) \\ - {}_2F_1\left(1 - \frac{\alpha+1}{\theta}, 1; q+2; 1-\lambda\right) \end{array} \right) \left. \right\} & , \quad 0 < \lambda < 1 \\ \frac{1}{\theta} \beta\left(\frac{1}{\theta}, q+1\right) - \frac{1}{\theta} \beta\left(\frac{\alpha+1}{\theta}, q+1\right) & , \quad \lambda = 1 \end{cases} \quad (2.14)
\end{aligned}$$

$$V_3 = \int_0^1 \left(\frac{x}{a}\right)^{pmt} dt = \begin{cases} \frac{\left(\frac{x}{a}\right)^{mp} - 1}{\ln\left(\frac{x}{a}\right)^{mp}} & , \quad x \neq a \\ 1 & , \quad \text{otherwise} \end{cases} \quad (2.15)$$

$$V_4 = \int_0^1 \left(\frac{x}{b}\right)^{pmt} dt = \begin{cases} \frac{\left(\frac{x}{b}\right)^{mp} - 1}{\ln\left(\frac{x}{b}\right)^{mp}} & , \quad x \neq b \\ 1 & , \quad \text{otherwise} \end{cases} \quad (2.16)$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2, property of the modulus, the Hölder inequality and by using (2.3), (2.4), (2.15) and (2.16) we have

$$\begin{aligned}
& |K_f(\lambda, \theta, x^m, a^m, b^m)| \leq m^{\theta+1} \\
& \times \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \left(\int_0^1 \left(\frac{x}{a}\right)^{pmt} dt \right)^{\frac{1}{p}} \left(\int_0^1 |t^\theta - \lambda|^q |f'(x^{mt} a^{m(1-t)})|^q dt \right)^{\frac{1}{q}} \right. \\
& \left. + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \left(\int_0^1 \left(\frac{x}{b}\right)^{pmt} dt \right)^{\frac{1}{p}} \left(\int_0^1 |t^\theta - \lambda|^q |f'(x^{mt} b^{m(1-t)})|^q dt \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq m^{\theta+1} \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} V_3^{\frac{1}{p}} \right. \\
&\quad \times \left(\int_0^1 |t^\theta - \lambda|^q \left[\frac{t^\alpha |f'(x^m)|^q}{+m(1-t^\alpha) |f'(a)|^q} \right] dt \right)^{\frac{1}{q}} \\
&\quad + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} V_4^{\frac{1}{p}} \\
&\quad \times \left(\int_0^1 |t^\theta - \lambda|^q \left[\frac{t^\alpha |f'(x^m)|^q}{+m(1-t^\alpha) |f'(b)|^q} \right] dt \right)^{\frac{1}{q}} \left. \right\} \quad (2.17)
\end{aligned}$$

By a simple computation we verify (2.13) and (2.14). If we use (2.13), (2.14), (2.15) and (2.16) in (2.17) we obtain (2.12). This completes the proof. \square

Corollary 14. *Under the assumptions of Theorem 8 with $x = \sqrt{ab}$, $\lambda = \frac{1}{3}$ from the inequality (2.12) we get the following Simpson type inequality for fractional integrals*

$$\begin{aligned}
&\frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left(\frac{1}{3}, \theta, (\sqrt{ab})^m, a^m, b^m \right) \right| = \left| \frac{1}{6} \left[f(a^m) + 4f((\sqrt{ab})^m) + f(b^m) \right] \right. \\
&\quad - \frac{2^{\theta-1} \Gamma(\theta+1)}{(m \ln \frac{b}{a})^\theta} \left[J_{(\sqrt{ab})^m}^\theta f(a^m) + J_{(\sqrt{ab})^m}^\theta f(b^m) \right] \left. \right| \leq \frac{m \ln \frac{b}{a}}{4} \\
&\quad \times \left\{ a^m \left(\frac{(\frac{b}{a})^{\frac{mp}{2}} - 1}{\ln(\frac{b}{a})^{\frac{mp}{2}}} \right)^{\frac{1}{p}} \left[\begin{array}{l} V_1(\theta, \frac{1}{3}, \alpha, q) |f'(x^m)|^q \\ + m V_2(\theta, \frac{1}{3}, \alpha, q) |f'(a)|^q \end{array} \right]^{\frac{1}{q}} \right. \\
&\quad \left. + b^m \left(\frac{(\frac{a}{b})^{\frac{mp}{2}} - 1}{\ln(\frac{a}{b})^{\frac{mp}{2}}} \right)^{\frac{1}{p}} \left[\begin{array}{l} V_1(\theta, \frac{1}{3}, \alpha, q) |f'(x^m)|^q \\ + m V_2(\theta, \frac{1}{3}, \alpha, q) |f'(b)|^q \end{array} \right]^{\frac{1}{q}} \right\}
\end{aligned}$$

Corollary 15. *Under the assumptions of Theorem 8 with $x = \sqrt{ab}$, $\lambda = 0$ from the inequality (2.12) we get the following midpoint-type inequality for fractional integrals*

$$\begin{aligned}
&\frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left(0, \theta, (\sqrt{ab})^m, a^m, b^m \right) \right| \\
&= \left| f((\sqrt{ab})^m) - \frac{2^{\theta-1} \Gamma(\theta+1)}{(m \ln \frac{b}{a})^\theta} \left[J_{(\sqrt{ab})^m}^\theta f(a^m) + J_{(\sqrt{ab})^m}^\theta f(b^m) \right] \right| \\
&\leq \frac{m \ln \frac{b}{a}}{4} \left\{ a^m \left(\frac{(\frac{b}{a})^{\frac{mp}{2}} - 1}{\ln(\frac{b}{a})^{\frac{mp}{2}}} \right)^{\frac{1}{p}} \left[\begin{array}{l} \frac{1}{\theta q + \alpha + 1} |f'(x^m)|^q \\ + \left(\frac{m}{\theta q + 1} - \frac{m}{\theta q + \alpha + 1} \right) |f'(a)|^q \end{array} \right]^{\frac{1}{q}} \right. \\
&\quad \left. + b^m \left(\frac{(\frac{a}{b})^{\frac{mp}{2}} - 1}{\ln(\frac{a}{b})^{\frac{mp}{2}}} \right)^{\frac{1}{p}} \left[\begin{array}{l} \frac{1}{\theta q + \alpha + 1} |f'(x^m)|^q \\ + \left(\frac{m}{\theta q + 1} - \frac{m}{\theta q + \alpha + 1} \right) |f'(b)|^q \end{array} \right]^{\frac{1}{q}} \right\}
\end{aligned}$$

Corollary 16. *Under the assumptions of Theorem 8 with $x = \sqrt{ab}$, $\lambda = 1$ from the inequality (2.12) we get the following trapezoid-type inequality for fractional*

integrals

$$\begin{aligned}
& \frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left(1, \theta, (\sqrt{ab})^m, a^m, b^m \right) \right| \\
&= \left| \frac{f(a^m) + f(b^m)}{2} - \frac{2^{\theta-1} \Gamma(\theta+1)}{(m \ln \frac{b}{a})^\theta} \left[J_{(\sqrt{ab})^m-}^\theta f(a^m) + J_{(\sqrt{ab})^m+}^\theta f(b^m) \right] \right| \\
&\leq \frac{m \ln \frac{b}{a}}{4} \left\{ a^m \left(\frac{(\frac{b}{a})^{\frac{mp}{2}} - 1}{\ln(\frac{b}{a})^{\frac{mp}{2}}} \right)^{\frac{1}{p}} \left[+ \left(\frac{1}{\theta} \beta \left(\frac{1}{\theta}, q+1 \right) - \frac{1}{\theta} \beta \left(\frac{\alpha+1}{\theta}, q+1 \right) \right) |f'(a)|^q \right]^{\frac{1}{q}} \right. \\
&\quad \left. + b^m \left(\frac{(\frac{a}{b})^{\frac{mp}{2}} - 1}{\ln(\frac{a}{b})^{\frac{mp}{2}}} \right)^{\frac{1}{p}} \left[+ \left(\frac{1}{\theta} \beta \left(\frac{1}{\theta}, q+1 \right) - \frac{1}{\theta} \beta \left(\frac{\alpha+1}{\theta}, q+1 \right) \right) |f'(b)|^q \right]^{\frac{1}{q}} \right\}
\end{aligned}$$

Corollary 17. *Let the assumptions of Theorem 7 hold. If $|f'(u)| \leq M$ for all $u \in [a^m, b]$ and $\lambda = 0$, then from the inequality (2.12) we get the following Ostrowski type inequality for fractional integrals*

$$\begin{aligned}
& \left| \left[\left(\ln \frac{x}{a} \right)^\theta + \left(\ln \frac{b}{x} \right)^\theta \right] f(x^m) - \frac{\Gamma(\theta+1)}{m^\theta} [J_{x^m-}^\theta f(a^m) + J_{x^m+}^\theta f(b^m)] \right| \\
&\leq mM \left[+ \left(\frac{1}{\theta q + \alpha + 1} \right) \right]^{\frac{1}{q}} \left\{ a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \left(\frac{(\frac{x}{a})^{mp} - 1}{\ln(\frac{x}{a})^{mp}} \right)^{\frac{1}{p}} \right. \\
&\quad \left. + b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \left(\frac{(\frac{x}{b})^{mp} - 1}{\ln(\frac{x}{b})^{mp}} \right)^{\frac{1}{p}} \right\}
\end{aligned}$$

for each $x \in [a, b]$

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